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A NOTE ON CHARACTERISTIC FUNCTIONS WHICH VANISH IDENTICALLY IN AN INTERVAL

by

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September, 1961

This research was supported by the Office of Naval Research under Contract No. Nonr-855(09) for research in probability and statistics at Chapel Hill. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Institute of Statistics
Mimeograph Series No. 306

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Some years ago, in connection with some unpublished work in the theory of queues, the question arose as to whether the characteristic function of a non-negative random variable could vanish identically in an interval. The purpose of this note is to show that such a thing is impossible.

There are, of course, characteristic functions which vanish identically in intervals. In fact the probability density function

$$f(x) = \frac{2 \sin^2 (x/2)}{\pi \ x^2} \quad , \quad -\infty < x < +\infty \ ,$$

is associated with the characteristic function

$$\phi(\theta) = \begin{cases} 1 - |\theta| & , \quad |\theta| \leq 1, \\ 0 & , \quad |\theta| > 1, \end{cases}$$

which vanishes in semi-infinite intervals. This characteristic function, however, is associated with a distribution over both positive and negative values. It is not clear what can be achieved for distributions over positive values only, although the following easily proved theorem gives some indications.

<u>Theorem 1. If $0 < \theta_1 < \theta_2 < \theta_3 < \ldots$ is an increasing sequence of angles, and if $k_1, k_2, k_3, \ldots$ is a sequence of positive integers, and if</u>

(1) $$\underline{\sum_{n=1}^{\infty} \frac{k_n}{\theta_n} < \infty ,}$$

<u>then there is a non-negative random variable whose characteristic function vanishes at $\theta_1, \theta_2, \theta_3, \ldots$, the zero at θ_n being k_n-fold, for all n.</u>

<u>Proof</u>. Let X_n be a random variable associated with the probability density function

$$f_n(x) = \frac{\theta_n}{2\pi} \quad \text{for} \quad 0 \leq x \leq \frac{2\pi}{\theta_n} ,$$

$$= 0 \qquad \text{otherwise.}$$

Let $X_n', X_n'', \ldots, X_n^{(k_n)}$ be k_n independent random variables distributed like X_n, and write $Z_n = X_n' + X_n'' + \ldots + X_n^{(k_n)}$. Then

(2) $$\psi_n(\theta) \equiv \mathcal{E} e^{i\theta Z_n} = e^{i\theta k_n \pi/\theta_n} \left\{ \frac{\sin^{k_n} (\pi\, \theta/k_n)}{(\pi\, \theta/\theta_n)^{k_n}} \right\} ,$$

and

(3) $$\mathcal{E} Z_n = \frac{\pi k_n}{\theta_n} ,$$

(4) $$\operatorname{Var} Z_n = \frac{k_n \pi^2}{3\, \theta_n^2} .$$

Furthermore, in view of (1), the random variables $\{ Z_n \}$ are uniformly bounded. Thus if $Z_1, Z_2, Z_3, \ldots$ are independent, the series

$$S = \sum_{j=1}^{\infty} Z_j$$

is almost certainly convergent, in virtue of the three series theorem and

(1), (3), and (4). The characteristic function of S is evidently

$$(5) \qquad \phi(\theta) = \prod_{n=1}^{\infty} \psi_n(\theta) \quad ,$$

where $\psi_n(\theta)$ is given in (2). Since S is non-negative and $\phi(\theta)$ obviously has a k_n-fold zero at θ_n, for all n, the theorem is proved.

We do not for a moment suppose that Theorem 1 is the strongest possible result of its kind that can be proved. It merely provides, as we have said, some indication of the distribution of zeros which can occur. It would naturally be of interest to obtain critical results, but we have been unable to obtain such. Nevertheless the following theorem shows that the characteristic function of a non-negative random variable has some restriction on the location of its zeros.

<u>Theorem 2. If $\phi(\theta)$ is the characteristic function of a non-negative random variable then $\phi(\theta)$ cannot vanish identically in an interval.</u>

<u>Proof</u>:- Let $F(x)$ be the distribution function of the non-negative random variable and write

$$(6) \qquad \Psi(u + iv) = \int_0^{\infty} e^{i(u+iv)x} \, dF(x) \quad ,$$

$$(7) \qquad \Phi(u + iv) = \frac{\Psi(u + iv)}{1 - i(u + iv)} \quad .$$

Evidently $\Phi(u + iv)$ is regular for $v > 0$ and, since $|\Psi(u + iv)| \leq 1$ for $v > 0$,

$$\int_{-\infty}^{+\infty} | \Phi (u + iv) |^2 \, du$$

exists and is bounded for every positive v.

Thus, for $v > 0$, by Titchmarsh (1948, p. 125), we have, for $I\, z > 0$,

$$(8) \qquad \Phi (z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\Phi (u)}{u - z} \, du ,$$

where, in the present situation, $\Phi (u)$ is the ordinary limit as $v \to 0+$ of $\Phi (u + iv)$. Thus

$$\Phi (u) = \frac{\psi (u)}{1 - iu} = \frac{1}{1 - iu} \int_{0}^{\infty} e^{iux} \, dF (x) .$$

Now suppose the characteristic function $\psi(u)$ vanishes identically in (a,b), $a < b$. Then (8) may be rewritten, for $I\, z > 0$,

$$(9) \qquad \Phi(z) = \frac{1}{2\pi i} \left(\int_{-\infty}^{a} + \int_{b}^{+\infty} \right) \frac{\Phi (u)}{u - z} \, d\, u .$$

The right side of (9) is a regular function of z in the upper half-plane. It is not hard to see that this regular function can be analytically continued, by a well-known expansion method, through the segment (a,b) on the real axis into the lower half-plane. Thus $\Phi(z)$ is regular in an open domain which contains a limit point of zeros. This implies that $\Phi(z)$ vanishes identically, and thus provides a contradiction. We may therefore conclude that $\Phi(u)$ cannot vanish identically in an interval.

We point out that an alternative proof of Theorem 2 can be deduced from a result of Bieberbach (1931 , p. 156) concerning functions analytic in a circle.

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